

SUPER-WAVELETS VERSUS POLY-BERGMAN SPACES

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ABSTRACT. Motivated by potential applications in multiplexing and by recent results on Gabor analysis with Hermite windows due to Gröchenig and Lyubarskii, we investigate vector-valued wavelet transforms and vector-valued wavelet frames, which constitute special cases of super-wavelets, with a particular attention to the case when the analyzing wavelet vector is constituted by functions Φ_n such that $\mathcal{F}\Phi_n(t) = t^{\frac{1}{2}}\ell_n(2t)$, where ℓ_n is a Laguerre function. We construct an isometric isomorphism between $L^2(\mathbb{R}^+, \mathbb{C}^n)$ and poly-Bergman spaces, with a view to relate the sampling sequences in the poly-Bergman spaces to the wavelet frames and super-frames with the windows Φ_n . One of the applications of the theory is a proof that $b \ln a < 2\pi(n+1)$ is a necessary condition for the (scalar) wavelet frame associated to the Φ_n to exist. This is the first known result of this type outside the setting of analytic functions (the case $n = 0$, which has been completely studied by Seip in 1993).

1. INTRODUCTION

1.1. Motivation. Over the recent years, an increasing attention has been given to the study of vector-valued versions of time-frequency and time-scale methods, driven in part by potential applications in *multiplexing*, an important method in telecommunications, computer networks and digital video.

Roughly speaking, "multiplexing" means: encoding n independent signals f_k as a single signal \mathbf{f} which contains all the information of all of the signals f_k .

Of course, the signals can always be combined into a single one by simple superposition; what makes the problem nontrivial, from a mathematical point of view, is to assure that the superposition is done in such a way that, when required, each of the signals f_k can be recovered from the multiplexed signal \mathbf{f} (*demultiplexing*). A recent mathematical approach to this problem has been proposed by means of the theory of *frames*. A sequence of functions $\{e_j\}_{j \in I}$ is said to be a frame in a Hilbert space H if there exist constants A and B such that, for every $f \in H$,

$$(1.1) \quad A \|f\|_H^2 \leq \sum_{j \in I} |\langle f, e_j \rangle|^2 \leq B \|f\|_H^2.$$

In [4], Balan has given an interpretation of standard multiplexing methods from the point of view of *super-frames*, which are vector-valued versions of frames (where the Hilbert space H is $L^2(\mathbb{R}, \mathbb{C}^n)$). The concept of super-frames has been investigated from the perspective of functional analysis by Han and Larson in [17] and several results have been obtained concerning the possibility of extending a frame to a

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super-frame [13], something that, not surprisingly, usually requires oversampling by a rate equal to dimension n , which represents the number of multiplexed signals [10], [16].

Within super-frames, super-wavelets have received a particular attention in [17], [13] and [10]. Alternative approaches to the construction of vector-valued wavelets have also been considered recently [7], [6]. In this paper we will provide yet another approach to the construction of vector-valued wavelets with emphasis in important special cases where, perhaps in a surprising manner, a new connection with complex analysis is revealed.

To better give a context to our work, let us review two fundamental ideas which have arisen recently in different mathematical communities.

The first fundamental idea comes from Gabor and complex analysis: if one considers the super-frames built from shifting and modulating a window with n Hermite functions, we are lead to a very structured situation, where an intriguing connection to complex analysis has been explored by Gröchenig and Lyubarskii in [16] and then in [1], where it is also shown that the *vector valued* Gabor representations with vectorial Hermite windows correspond to *polyanalytic* functions (we provide definitions some lines below), in much the same way the scalar Gabor representations with Gaussian correspond to analytic functions.

The second fundamental idea is implicit in the link between the results of Hutnik [20], [21] and of Vasilevski [31], given the intriguing ressemblance between the description of *wavelet spaces* and *poly-Bergman spaces*.

In the present paper, we describe a connection between vector-valued wavelets and polyanalytic functions, in analogy to the one existing for the Gabor case [1], [2]. Indeed, polyanalytic Bergman spaces are the function-theoretic analogues of wavelet spaces, when one chooses as analysing wavelet a certain vector defined in terms of Laguerre functions. We will use this setting to provide a theoretical solution of the problem of multiplexing and demultiplexing vectorial signals in $L^2(\mathbb{R}^+, \mathbb{C}^n)$. We remind that, by the Paley-Wiener theorem, this space is unitarily equivalent to the vectorial Hardy space on the upper-half plane, $H^2(\mathbf{U}, \mathbb{C}^n)$.

1.2. Description of the results. To describe our results we need some definitions.

Let $\mathbf{U} = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ stand for the upper half plane. The *polyanalytic Bergman space*, $\mathbf{A}^n(\mathbf{U})$ (poly-Bergman space, for short), is constituted by the complex valued functions defined on the upper half plane and such that

$$\left(\frac{d}{d\bar{z}}\right)^n f(z) = 0 \text{ and } \int_{\mathbf{U}} |f(z)|^2 dz < \infty.$$

The space $\mathbf{A}^0(\mathbf{U}) = A(\mathbf{U})$ is the usual (analytic) Bergman space in the upper half plane.

The potential applications of polyanalytic Bergman spaces in multiplexing are suggested by the orthogonal decomposition of the space $\mathbf{A}^n(\mathbf{U})$ in subspaces called *true polyanalytic Bergman spaces*, which will be denoted by $\mathcal{A}^k(\mathbf{U})$:

$$\mathbf{A}^n(\mathbf{U}) = \mathcal{A}^0(\mathbf{U}) \oplus \dots \oplus \mathcal{A}^{n-1}(\mathbf{U}).$$

In face of this decomposition, if we start with a vector $\mathbf{f} = (f_0, \dots, f_{n-1}) \in L^2(\mathbb{R}^+, \mathbb{C}^n)$ and find a unitary transform (call it *Ber* ^{n}) such that

$$f_k \in L^2(\mathbb{R}^+) \xrightarrow{\text{Ber}^k} F_k \in \mathcal{A}^k(\mathbf{U}),$$

then we can store all the F_k 's into a single multiplexed function

$$F = F_0 + \dots + F_{n-1} \in \mathbf{A}^n(\mathbf{U}).$$

Then, thanks to the orthogonal decomposition of $\mathbf{A}^n(\mathbf{U})$, it is possible to recover each of the F_k , projecting the multiplexed signal F into $\mathcal{A}^k(\mathbf{U})$. This is possible to be done using the reproducing kernel $K^k(z, w)$ of the spaces $\mathcal{A}^k(\mathbf{U})$: Given $F \in \mathbf{A}^n(\mathbf{U})$, its true polyanalytic component $F_k \in \mathcal{A}^k(\mathbf{U})$ can be recovered by the orthogonal projection of F over the space $\mathcal{A}^k(\mathbf{U})$:

$$F_k(z) = \langle F(w), K^k(z, w) \rangle_{\mathbf{A}^n(\mathbf{U})}.$$

One of the main goals of the present paper is the construction of the unitary transform $Ber^n : L^2(\mathbb{R}^+) \rightarrow \mathcal{A}^{n-1}(\mathbf{U})$ mentioned above. We will call it the *true polyanalytic Bergman transform* and define it in terms of the (analytic) Bergman transform

$$Ber f(z) = \int_0^\infty t^{\frac{1}{2}} f(t) e^{izt} dt,$$

which is unitary $Ber : L^2(\mathbb{R}^+) \rightarrow A(\mathbf{U})$. Starting from this, we introduce the *true polyanalytic Bergman transform*

$$Ber^n f(z) = \frac{1}{(2i)^n n!} \left(\frac{d}{dz} \right)^n [s^n Ber f(z)].$$

It is at this point that the connections to wavelets shows up. Actually, the key in proving the unitarity of Ber^n is to recognize that, defining the functions Φ_n as

$$\mathcal{F}\Phi_n(t) = t^{\frac{1}{2}} \ell_n^0(2t),$$

where ℓ_n^0 is a Laguerre function (see section 2 for definition), the true polyanalytic Bergman transform is related to the wavelet transform by the formula

$$Ber^n f(z) = s^{-1} W_{\Phi_n}(\mathcal{F}^{-1} f)(x, s).$$

The next goal is to define a unitary isomorphism $\mathbf{Ber}^n : L^2(\mathbb{R}^+, \mathbf{C}^n) \rightarrow \mathbf{A}^n(\mathbf{U})$. For $\mathbf{f} \in L^2(\mathbb{R}^+, \mathbf{C}^n)$, the *polyanalytic Bergman transform* is

$$\mathbf{Ber}^n \mathbf{f} = \sum_{0 \leq k \leq n-1} Ber^k f_k.$$

This provides a unitary operator $\mathbf{Ber}^n : L^2(\mathbb{R}^+, \mathbf{C}^n) \rightarrow \mathbf{A}^n(\mathbf{U})$, which relates vector-valued functions to polyanalytic functions. Here vector-valued wavelet transforms enter the picture. We will interpretate the polyanalytic Bergman transform is a special case of a general vector valued wavelet transform defined as

$$\mathbf{W}_g \mathbf{f}(x, s) = \sum_{0 \leq k \leq n-1} W_{g_k} f_k(x, s),$$

where $W_{g_k} f_k$ stands for the scalar wavelet transform of the k th component of the vector $\mathbf{f} = (f_0, \dots, f_{n-1})$, analyzed with the wavelet g_k , which is a component of the vector-valued analyzing wavelet \mathbf{g} . The components of the vector $\mathbf{g} = (g_0, \dots, g_{n-1})$ are selected in such a way their Fourier transforms satisfy

$$\langle \mathcal{F}g_i, \mathcal{F}g_j \rangle_{L^2(\mathbb{R}^+, t^{-1})} = \delta_{i,j}.$$

In particular, one can choose the vector-valued analyzing wavelet $\Phi_n = (\Phi_0, \dots, \Phi_{n-1})$. In doing so, we arrive at the relation between the vector-valued wavelet transform and the polyanalytic Bergman transform:

$$\mathbf{W}_{\Phi_n}(\mathcal{F}^{-1}\mathbf{f})(x, s) = s\mathbf{Ber}^n\mathbf{f}.$$

Then, we compute explicitly the reproducing kernels of the spaces $\mathcal{A}^k(\mathbf{U})$, required for the orthogonal projections. They are given by the formula

$$K^n(z, w) = \frac{1}{(2i)^n n!} \left(\frac{d}{dz} \right)^n \left[s^n \Omega_n \left(\frac{z+w}{\eta} \right) \right],$$

where

$$\Omega_n(z) = 4(n+z-i) \left(\frac{1}{z+i} \right)^3 \left(\frac{z-i}{z+i} \right)^{n-1}.$$

In the final section we study sampling sequences in polyanalytic Bergman spaces. This is equivalent to the study of wavelet frames with the functions Φ_n . The section contains a contribution to the subject of *affine density*, the study of the density of sequences Λ yielding inequalities of the form (wavelet frames):

$$A \|f\|_{H^2(\mathbf{U})}^2 \leq \sum_{(x,s) \in \Lambda} |\langle f, T_x D_s \psi \rangle|^2 \leq B \|f\|_{H^2(\mathbf{U})}^2.$$

There has been a considerable research activity around the topic of affine density, and currently there are three different approaches (see [18], [19], [22], [30], [29] and the monograph [23]). The definitions of density in Seip [29] and [18], [19], [22], agree in the case of the hyperbolic lattice $\Lambda(a, b) = \{(a^m b k, a^m)\}$, where the density is $1/b \log a$, up to constants independent of a and b .

It is straightforward to see that there is no universal necessary lower bound for an arbitrary function to generate a frame associated to $\Lambda(a, b)$, in contrast to the situation in Gabor analysis [26], [27]. In the literature we have found only one example where such a bound exists. It is the case of the wavelet defined on the Fourier side by $\mathcal{F}\psi_\alpha(t) = \mathbf{1}_{[0, \infty]} t^\alpha e^{-t}$, where the problem can be translated into a sampling problem in Bergman spaces of analytic functions, which has been completely understood in [29].

Thanks to the connection to polyanalytic Bergman spaces, we will provide a family of examples (which in the case $n = 0$ reduces to ψ_α) where an explicit bound on the constant $b \log a$ is shown to be necessary, providing thus the first results in this direction since the sharp conditions associated to the Poisson wavelet. We obtain such conditions by investigating the sampling sequences in the true polyanalytic Bergman space, $\mathcal{A}^n(\mathbf{U})$. A sampling sequence in $\mathcal{A}^n(\mathbf{U})$ is one originating inequalities of the form

$$A \|F\|_{\mathcal{A}^n(\mathbf{U})}^2 \leq \sum_{z \in \Gamma(a, b)} |F(z)|^2 \leq B \|F\|_{\mathcal{A}^n(\mathbf{U})}^2.$$

Thanks to the identity

$$\langle f, T_x D_s \Phi_n \rangle_{H^2(\mathbf{U})} = s \mathbf{Ber}^n \mathcal{F}f(z),$$

this inequality is equivalent to the wavelet frame inequality. To prove the existence of sampling sequences, we use some results from Ascensi and Bruna [3]. Finally, we combine an argument used by Seip [28] with a result from [3] in order to prove a necessary condition for the existence of sampling sequences in $\mathcal{A}^n(\mathbf{U})$. In terms

of wavelet frames, such condition requires that, if $\mathcal{W}(\Phi_n, \Lambda)$ is a wavelet frame for $H^2(\mathbf{U})$, then

$$b \log a < 2\pi(n+1).$$

1.3. Organization of the paper. The outline is as follows. We have a background section where we review some known facts concerning wavelets, Laguerre functions and the fundamental facts concerning analytic and polyanalytic Bergman spaces and the connection between Bergman spaces and wavelets provided by the analytic Bergman transform. Then, in the third section, we define a vector-valued version of the continuous wavelet transform and some of its elementary properties. The fourth section is the most important of the paper. We introduce the true polyanalytic and the polyanalytic transforms, and describe their relation with wavelets and vector-valued wavelets. Such a relation is fundamental in order to prove the unitarity of the polyanalytic transforms. In section 6, the structure of the polyanalytic Bergman spaces is investigated using these new tools. We obtain a sequence of rational functions which is orthogonal in the upper half plane, defined by its Rodrigues formula. This sequence of rational functions is a basis of the polyanalytic Bergman space. Then we obtain an explicit formula for the reproducing kernel of the polyanalytic Bergman spaces, in the form of a differential operator which also resembles a Rodrigues formula. In the last section of the paper we present some results concerning sampling sequences in spaces of polyanalytic functions and their consequences in terms of wavelet frames.

2. BACKGROUND

2.1. The wavelet transform. For every $x \in \mathbb{R}$ and $s \in \mathbb{R}^+$ define the operators translation, modulation and dilation as

$$T_x f(t) = f(t - x); \quad M_x f(t) = e^{-ixt} f(t),$$

and

$$D_s f(t) = s^{-\frac{1}{2}} f(s^{-1}t).$$

Fix a function $g \neq 0$. Then the continuous wavelet transform of a function f with respect to a wavelet g is defined, for every $x \in \mathbb{R}$, $s > 0$ as

$$(2.1) \quad W_g f(x, s) = \langle f, T_x D_s g \rangle_{L^2(\mathbb{R})}.$$

The following relations are usually called *the orthogonal relations for the wavelet transform*. Assume that $g_1, g_2 \in L^2(\mathbb{R})$ satisfy

$$\langle \mathcal{F}g_1, \mathcal{F}g_2 \rangle_{L^2(\mathbb{R}^+, t^{-1})} < \infty.$$

Then, for all $f_1, f_2 \in L^2(\mathbb{R})$,

$$(2.2) \quad \langle W_{g_1} f_1, W_{g_2} f_2 \rangle_{L^2(\mathbf{U}, s^{-2} dx ds)} = \langle \mathcal{F}g_1, \mathcal{F}g_2 \rangle_{L^2(\mathbb{R}^+, t^{-1})} \langle f_1, f_2 \rangle_{L^2(\mathbb{R})}.$$

A function $g \in L^2(\mathbb{R})$ is said to be admissible if

$$\|\mathcal{F}g\|_{L^2(\mathbb{R}^+, t^{-1})}^2 = K,$$

where K is a constant. If g is admissible, then for all $f \in L^2(\mathbb{R})$ we have

$$(2.3) \quad \|W_g f\|_{L^2(\mathbf{U}, s^{-2} dx ds)}^2 = K \|f\|_{L^2(\mathbb{R})}^2.$$

Therefore, the continuous wavelet transform provides an isometric inclusion, being an isometry when $K = 1$.

$$W_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbf{U}, s^{-2} dx ds),$$

If we restrict to functions $f \in H^2(\mathbf{U})$, the Hardy space in the upper half plane constituted by analytic functions f such that

$$\sup_{0 < s < \infty} \int_{-\infty}^{\infty} |f(x + is)|^2 dx < \infty.$$

By the Paley-Wiener theorem, $H^2(\mathbf{U})$ is constituted by the functions whose Fourier transform,

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx,$$

is supported in \mathbb{R}^+ and belongs to $L^2(\mathbb{R}^+)$. Using the action of the Fourier transform on the dilation and translation operators,

$$\mathcal{F}D_s f = D_{\frac{1}{s}} \mathcal{F}f \quad \text{and} \quad \mathcal{F}T_x f = M_{-x} \mathcal{F}f,$$

we can use Plancherel theorem to rewrite the wavelet transform "on the Fourier side" as

$$W_g f(x, s) = \left\langle \mathcal{F}f, M_{-x} D_{\frac{1}{s}} \mathcal{F}g \right\rangle_{L^2(\mathbb{R}^+)}.$$

2.2. The Laguerre functions. The Laguerre polynomials will play a central role in our discussion. One way to define them is by the power series

$$(2.4) \quad L_n^\alpha(x) = \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(\alpha + 1)_k} \frac{x^k}{k!}.$$

This is equivalent to the Rodrigues formula

$$(2.5) \quad L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n} [e^{-x} x^{\alpha+n}].$$

The Laguerre functions are defined as

$$l_n^\alpha(x) = \mathbf{1}_{[0, \infty)}(x) e^{-x/2} x^{\alpha/2} L_n^\alpha(x).$$

It is well known that, for $\alpha \geq 0$, these functions constitute an orthogonal basis for the space $L^2(0, \infty)$.

3. BERGMAN SPACES

3.1. Analytic and polyanalytic Bergman spaces. With the Wirtinger differential operator notation,

$$\frac{d}{dz} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial s} \right), \quad \frac{d}{d\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial s} \right),$$

a complex valued function f , is said to be analytic in a domain, if, for every z in such domain, it satisfies

$$\frac{d}{d\bar{z}} f(z) = 0.$$

More generally, f is said to be polyanalytic of order n if

$$\left(\frac{d}{d\bar{z}} \right)^n f(z) = 0.$$

Then $A(\mathbf{U})$ stands for the *Bergman space* in the upper half plane, constituted by the analytic functions in \mathbf{U} such that

$$(3.1) \quad \int_{\mathbf{U}} |f(z)|^2 dx ds < \infty.$$

The space constituted by the polyanalytic functions of order n , equipped with the same norm as the Bergman space is called the *polyanalytic Bergman space*, $\mathbf{A}^n(\mathbf{U})$. With this notation, $\mathbf{A}^1(\mathbf{U}) = A(\mathbf{U})$. Consider also the *true polyanalytic Bergman space*, $\mathcal{A}^n(\mathbf{U})$, defined as

$$\mathcal{A}^{n-1}(\mathbf{U}) = \mathbf{A}^n(\mathbf{U}) \ominus \mathbf{A}^{n-1}(\mathbf{U}),$$

so that the following decomposition holds:

$$(3.2) \quad \mathbf{A}^n(\mathbf{U}) = \mathcal{A}^0(\mathbf{U}) \oplus \dots \oplus \mathcal{A}^{n-1}(\mathbf{U}).$$

3.2. The Bergman transform. We can relate the wavelet transform to Bergman spaces of analytic functions, by choosing the window ψ_α such that

$$(3.3) \quad \mathcal{F}\psi_\alpha(t) = \mathbf{1}_{[0,\infty]} t^\alpha e^{-t}.$$

Writing $z = x + si$ gives

$$(3.4) \quad \mathcal{F}T_{-x}D_s\psi_\alpha(t) = \mathbf{1}_{[0,\infty]} s^{\alpha+\frac{1}{2}} t^\alpha e^{izt}$$

and

$$(3.5) \quad W_{\psi_\alpha} f(-x, s) = s^{\alpha+\frac{1}{2}} \int_0^\infty t^\alpha (\mathcal{F}f)(t) e^{izt} dt.$$

Considering $f \in H^2(\mathbf{U})$, then $\mathcal{F}f \in L^2(\mathbb{R}^+)$. This motivates the definition of the *Bergman transform* of order α as the analytic part of (3.5):

$$(3.6) \quad Ber_\alpha f(z) = s^{-\alpha} W_{\psi_{\alpha-\frac{1}{2}}}(\mathcal{F}^{-1}f)(-x, s) = \int_0^\infty t^{\alpha-\frac{1}{2}} f(t) e^{izt} dt.,$$

We will write

$$(3.7) \quad Ber f(z) = Ber_1 f(z) = \int_0^\infty t^{\frac{1}{2}} f(t) e^{izt} dt,$$

in order to obtain an isometric transformation

$$Ber : L^2(\mathbb{R}^+) \rightarrow A(\mathbf{U}).$$

4. A CONTINUOUS VECTOR VALUED WAVELET TRANSFORM

Now, consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbf{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, \dots, f_{n-1})$ with the inner product

$$(4.1) \quad \langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}} = \sum_{0 \leq k \leq n-1} \langle f_k, g_k \rangle_{L^2(\mathbb{R})}.$$

Definition 1. Let $\mathbf{g} = (g_0, \dots, g_{n-1})$ be a vector of functions in \mathcal{H} such that

$$(4.2) \quad \langle \mathcal{F}g_i, \mathcal{F}g_j \rangle_{L^2(\mathbb{R}^+, t^{-1})} = \delta_{i,j}$$

The continuous vector valued wavelet transform of a function $\mathbf{f} = (f_1, \dots, f_{n-1})$ with respect to the vectorial window \mathbf{g} is defined, for every $x \in \mathbb{R}, s \in \mathbb{R}^+$, as

$$(4.3) \quad \mathbf{W}_{\mathbf{g}} \mathbf{f}(x, \omega) = \langle \mathbf{f}, D_s T_x \mathbf{g} \rangle_{\mathcal{H}}.$$

We can also write

$$\mathbf{W}_{\mathbf{g}}\mathbf{f}(x, s) = \sum_{0 \leq k \leq n-1} W_{g_k} f_k(x, s).$$

This defines a map

$$\mathbf{W}_{\mathbf{g}}\mathbf{f} : \mathcal{H} \rightarrow L^2(\mathbf{U}, s^{-2} ds ds).$$

The orthogonality condition imposed on the vector \mathbf{g} allows the superwavelet transform to retain most of the properties of the scalar Wavelet transform. In particular, we have vector valued versions of the isometric property and orthogonality relations.

Proposition 1. *Let \mathbf{g} satisfy (4.2). Then, for $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{H}$,*

$$(4.4) \quad \langle \mathbf{W}_{\mathbf{g}}\mathbf{f}_1, \mathbf{W}_{\mathbf{g}}\mathbf{f}_2 \rangle_{L^2(\mathbf{U}, s^{-2} dz)} = \langle \mathbf{f}_1, \mathbf{f}_2 \rangle_{\mathcal{H}}.$$

In particular, $\mathbf{W}_{\mathbf{g}}\mathbf{f}$ is an isometry between Hilbert spaces, that is

$$(4.5) \quad \|\mathbf{W}_{\mathbf{g}}\mathbf{f}\|_{L^2(\mathbf{U}, s^{-2} dz)} = \|\mathbf{f}\|_{\mathcal{H}}.$$

Proof. From (2.2) and (4.2),

$$(4.6) \quad \langle W_{g_k} f_k, W_{g_j} f_j \rangle_{L^2(\mathbf{U}, s^{-2} dx ds)} = \langle f_k, f_j \rangle_{L^2(\mathbb{R})} \times \delta_{k,j}.$$

Then,

$$\begin{aligned} \langle \mathbf{W}_{\mathbf{g}}\mathbf{f}_1, \mathbf{W}_{\mathbf{g}}\mathbf{f}_2 \rangle_{L^2(\mathbf{U}, s^{-2} dx ds)} &= \sum_{0 \leq k, j \leq n-1} \langle W_{g_k} f_{1,k}, W_{g_j} f_{2,k} \rangle_{L^2(\mathbf{U}, s^{-2} dx ds)} \\ &= \sum_{0 \leq k, j \leq n-1} \langle f_{1,k}, f_{2,j} \rangle_{L^2(\mathbb{R})} \times \delta_{k,j} \\ &= \sum_{0 \leq k \leq n-1} \langle f_{1,k}, f_{2,k} \rangle_{L^2(\mathbb{R})} \\ &= \langle \mathbf{f}_1, \mathbf{f}_2 \rangle_{\mathcal{H}}. \end{aligned}$$

□

Now let $\mathbf{W}_{\mathbf{g}}$ stand for the subspace of $L^2(\mathbb{R}^2)$ constituted by the image of \mathcal{H} under the vector valued wavelet transform $\mathbf{W}_{\mathbf{g}}\mathbf{f}$:

$$\mathbf{W}_{\mathbf{g}} = \{\mathbf{W}_{\mathbf{g}}\mathbf{f} : \mathbf{f} \in \mathcal{H}\}.$$

Since

$$\mathbf{W}_{\mathbf{g}}\mathbf{f} = \sum_{0 \leq k \leq n-1} W_{g_k} f_k$$

and

$$\langle W_{g_k} f_k, W_{g_j} f_j \rangle_{L^2(\mathbf{U}, s^{-2} dx ds)} = \delta_{k,j},$$

we know that every $F \in \mathbf{W}_{\mathbf{g}}$ can be written in a unique way in the form

$$(4.7) \quad F = F_0 + \dots + F_{n-1}.$$

As a result,

$$(4.8) \quad \mathbf{W}_{\mathbf{g}} = \mathcal{W}_{g_0} \oplus \dots \oplus \mathcal{W}_{g_{n-1}},$$

where

$$\mathcal{W}_{g_j} = \{W_{g_j} f : f \in L^2(\mathbb{R})\}.$$

Proposition 2. *The space $\mathbf{W}_{\mathbf{g}}$ is a Hilbert space with reproducing kernel given by*

$$(4.9) \quad \mathbf{k}(z, w) = \langle T_{\eta} D_u \mathbf{g}, T_x D_s \mathbf{g} \rangle_{\mathcal{H}} = \sum_{j=0}^{n-1} k_j(z, w),$$

where $k_j(z, w)$ is the reproducing kernel of \mathcal{W}_{g_j} .

Proof. Let $\mathbf{F} \in \mathbf{W}_{\mathbf{g}}$. There exists $\mathbf{f} \in \mathcal{H}$ such that $\mathbf{F} = \mathbf{W}_{\mathbf{g}} \mathbf{f}$. By definition, $\mathbf{k}(z, \cdot) = \mathbf{W}_g(T_x D_s \mathbf{g})$. Thus, using (4.4),

$$\begin{aligned} \langle \mathbf{F}, \mathbf{k}(z, \cdot) \rangle_{L^2(\mathbf{U}, s^{-2} dz)} &= \langle \mathbf{W}_{\mathbf{g}} \mathbf{f}, \mathbf{W}_g(T_x D_s \mathbf{g}) \rangle_{L^2(\mathbf{U}, s^{-2} dz)} \\ &= \langle \mathbf{f}, T_x D_s \mathbf{g} \rangle_{\mathcal{H}} \\ &= \mathbf{F}(z). \end{aligned}$$

The second inequality follows from the well known fact that the reproducing kernel of the space \mathcal{W}_{g_j} is given by $\langle T_{\eta} D_u g_j, T_x D_s g_j \rangle_{L^2(\mathbb{R})}$. \square

Through the paper, we will restrict ourselves to vectors \mathbf{f} such that the Fourier transform of each of its components is supported in \mathbb{R}^+ and belongs to $L^2(\mathbb{R}^+)$. In such a case, $\mathcal{H} = H^2(\mathbf{U}, \mathbb{C}^n)$ and, on the Fourier side, the notation $\mathcal{H}^+ = L^2(\mathbb{R}^+, \mathbb{C}^n)$, will be used.

5. THE POLYANALYTIC BERGMAN TRANSFORM

In this section we will study a special case of the continuous vector valued wavelet transform, when the vector is defined in terms of Laguerre functions. First we treat the scalar case, which originates a unitary map onto the *true* polyanalytic Bergman space. We show that the required unitary mappings can be related to special wavelet transforms and, via a connection to the previous section, we define the vector valued polyanalytic transform onto the polyanalytic space (which we call the polyanalytic Bergman transform).

5.1. The true polyanalytic Bergman transform. First we will study the transform that allows, in the multiplexing context explained in the introduction, to send each signal $f_k \in L^2(\mathbb{R}^+)$ to a space $\mathcal{A}^k(\mathbf{U})$.

Definition 2. *The true polyanalytic Bergman transform of order n is the transform mapping every $f \in L^2(\mathbb{R}^+)$ to*

$$(5.1) \quad \text{Ber}^n f(z) = \frac{1}{(2i)^n n!} \left(\frac{d}{dz} \right)^n [s^n F(z)],$$

where $F = \text{Ber } f$ and $f \in L^2(\mathbb{R}^+)$.

The purpose of this section is to prove that the transform Ber^n is unitary

$$\text{Ber}^n : L^2(\mathbb{R}^+) \rightarrow \mathcal{A}^n(\mathbf{U}).$$

We will need some identities which have independent interest. First observe that, since

$$L_n^0(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!},$$

we have

$$(5.2) \quad \ell_n^0(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k!} \mathbf{1}_{[0, \infty]}(x) t^k e^{-t/2} = t^{-\frac{1}{2}} \sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{F} \psi_{k+\frac{1}{2}}\left(\frac{t}{2}\right).$$

Due to this observation, it is reasonable to expect that the functions Φ_n , defined by

$$(5.3) \quad \mathcal{F}\Phi_n(t) = t^{\frac{1}{2}} \ell_n^0(2t),$$

will play a distinguished role in our analysis. Indeed, an essential step in the proof of the unitary property is to write (5.1) in terms of a wavelet transform with analysing wavelet Φ_n .

Proposition 3. *The true polyanalytic Bergman transform of order n can be written as:*

(1) A polyanalytic function of order $n + 1$:

$$(5.4) \quad \text{Ber}^n f(z) = \frac{1}{(2i)^n n!} \sum_{k=0}^n (2i)^k \binom{n}{k} \frac{1}{k!} s^k F^{(k)}(z).$$

(2) In terms of analytic Bergman transforms of different orders:

$$(5.5) \quad \text{Ber}^n f(z) = \sum_{k=0}^n (-2)^k \binom{n}{k} s^k \text{Ber}_{k+1} f(z).$$

(3) In terms of a wavelet transform:

$$(5.6) \quad \text{Ber}^n f(z) = s^{-1} W_{\Phi_n}(\mathcal{F}^{-1}f)(x, s).$$

Proof. The first identity follows from a standard application of Leibnitz formula. Then, since

$$\frac{d}{d\bar{z}} s^k = \frac{1}{2} k s^{k-1},$$

we have

$$\left(\frac{d}{d\bar{z}} \right)^{n+1} \text{Ber}^n f(z) = 0,$$

and $\text{Ber}^n f$ is polyanalytic of order $n + 1$. To prove (5.5), observe that differentiating (3.7) under the integral sign gives

$$(5.7) \quad F^{(k)}(z) = \left(\frac{d}{dz} \right)^k \text{Ber} f(z) = i^k \text{Ber}_{k+1} f(z).$$

Applying this to (5.4) gives (5.5).

Now (5.6). Combining (5.2) with (5.3) and inverting the Fourier transform gives

$$\Phi_n(t) = \sum_{k=0}^n (-2)^k \binom{n}{k} \psi_{k+\frac{1}{2}}(t).$$

Then,

$$\begin{aligned} W_{\Phi_n} f(x, s) &= \sum_{k=0}^n (-2)^k \binom{n}{k} W_{\psi_{k+\frac{1}{2}}} f(-x, s) \\ &= \sum_{k=0}^n (-2)^k \binom{n}{k} s^{k+1} \text{Ber}_{k+1} \mathcal{F}f(z) \\ &= s \text{Ber}^n(\mathcal{F}f)(z). \end{aligned}$$

and (5.6) follows. \square

Now we can prove the main result. The idea consists in writing the wavelet transform (5.6) as a composition of several unitary operators and is suggested by the techniques used in [31] and [21]. We need to introduce two auxiliary operators. For convenience write $L^2(\mathbf{U}) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)$ and define the unitary operators $U_{1,2} : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)$:

$$\begin{aligned} U_1(F)(x, s) &= (\mathcal{F}^{-1} \otimes I)(F)(x, s) \\ U_2(F)(x, s) &= \frac{1}{\sqrt{2|x|}} F(x, \frac{s}{2|x|}). \end{aligned}$$

We will need the following result of Vasilevski [31].

Theorem A [31, Corollary 4.2] *Let L_n stand for the space generated by $1_{[0,\infty]} l_n^0$. The operator $U = U_2 U_1$,*

$$U : \mathcal{A}^n(\mathbf{U}) \rightarrow L^2(\mathbb{R}^+) \otimes L_n$$

such that, given $f \in \mathcal{A}^n(\mathbf{U})$,

$$(Uf)(x, s) = \mathbf{1}_{[0,\infty]}(x) f(x) l_n^0(s),$$

is unitary.

We now combine Theorem A with Proposition 3 to prove the main result.

Theorem 1. *The transform $\text{Ber}^n : L^2(\mathbb{R}^+) \rightarrow \mathcal{A}^n(\mathbf{U})$ is unitary.*

Proof. Consider the unitary operator

$$R_k : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+) \otimes L_k$$

defined by

$$(R_k f)(x, s) = \mathbf{1}_{[0,\infty]}(x) f(x) l_k^0(s).$$

Then the composition $U^{-1} R_k$ is also unitary

$$U^{-1} R_k : L^2(\mathbb{R}^+) \rightarrow \mathcal{A}^k(\mathbf{U})$$

We will now show that this transform is exactly Ber^n . From the definition of U_2 it is easy to see that

$$U_2^{-1}(F)(x, s) = \sqrt{2|x|} F(x, 2|x|s)$$

and

$$(U_2^{-1} R_k f)(x, s) = \mathbf{1}_{[0,\infty]}(x) \sqrt{2x} f(x) l_k^0(2xs).$$

Applying U_1^{-1} gives

$$\begin{aligned} (U^{-1} R_k f)(x, s) &= s^{-1} \int_0^\infty f(t) s^{\frac{1}{2}} (2ts)^{\frac{1}{2}} l_k^0(2ts) e^{ixt} dt \\ &= s^{-1} \int_{\mathbb{R}} (\mathcal{F}^{-1} f)(t) s^{-\frac{1}{2}} \overline{\Phi_n(s^{-1}(t-x))} dt \\ &= s^{-1} W_{\Phi_n}(\mathcal{F}^{-1} f)(x, s) \\ &= \text{Ber}^n f(z), \end{aligned}$$

by using the identity (5.6). □

5.2. The polyanalytic Bergman transform. Now, consider the Hilbert space $\mathcal{H}^+ = L^2(\mathbb{R}^+, \mathbb{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, \dots, f_{n-1})$ with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}^+} = \sum_{0 \leq k \leq n-1} \langle f_k, g_k \rangle_{L^2(\mathbb{R}^+)}.$$

Definition 3. *The polyanalytic Bergman transform of order n is defined, for $\mathbf{f} \in \mathcal{H}^+$ as*

$$\mathbf{Ber}^n \mathbf{f} = \sum_{0 \leq k \leq n-1} \mathbf{Ber}^k f_k.$$

If we take, then we have the following relation with the polyanalytic Bergman transform:

Theorem 2. *The polyanalytic Bergman transform of order n is a unitary operator*

$$\mathbf{Ber}^n : \mathcal{H}^+ \rightarrow \mathbf{A}^n(\mathbf{U})$$

Proof. To see that it is onto, let $\mathbf{F} \in \mathbf{A}^n(\mathbf{U})$. Then, using (3.2), write

$$\mathbf{F} = F_0 + \dots + F_{n-1},$$

with $F_k \in A^k(\mathbf{U})$, $k = 0, \dots, n-1$. Since \mathbf{Ber}^k is onto, for every $k = 0, \dots, n-1$ there exists $f_k \in L^2(\mathbb{R}^+)$ such that $F_k = \mathbf{Ber}^k f_k$. To prove the isometry, we first relate the polyanalytic Bergman transform to the vector valued wavelet transform with the vectorial window $\Phi_n = (\Phi_0, \dots, \Phi_{n-1})$, using the identity $\mathbf{Ber}^n f(z) = s^{-1} W_{\Phi_n}(\mathcal{F}^{-1} f)(x, s)$:

$$\mathbf{W}_{\Phi_n}(\mathcal{F}^{-1} \mathbf{f})(x, s) = \sum_{0 \leq k \leq n-1} W_{\Phi_k}(\mathcal{F}^{-1} f_k)(x, s) = \sum_{0 \leq k \leq n-1} s \mathbf{Ber}^k f_k = s \mathbf{Ber}^n \mathbf{f}.$$

Now, combining this with (4.5),

$$\|\mathbf{Ber}^n \mathbf{f}\|_{\mathbf{A}^n(\mathbf{U})} = \|\mathbf{W}_{\Phi_n}(\mathcal{F}^{-1} \mathbf{f})\|_{L^2(\mathbf{U}, s^{-2} dx ds)} = \|\mathcal{F}^{-1} \mathbf{f}\|_{\mathcal{H}} = \|\mathbf{f}\|_{\mathcal{H}^+}.$$

□

6. THE STRUCTURE OF POLYANALYTIC BERGMAN SPACES

The purpose of this section is to apply the connection to wavelet transforms to study polyanalytic Bergman spaces. We will obtain an orthogonal basis for $\mathbf{A}^n(\mathbf{U})$ and compute an explicit formula for the reproducing kernel. The reproducing kernel, $K^n(z, w)$, of the true polyanalytic Bergman space $\mathcal{A}^n(\mathbf{U})$ is also very important, since once we have a function $F \in \mathbf{A}^n(\mathbf{U})$, we can recover its true polyanalytic component $F_k \in A^k(\mathbf{U})$ by the orthogonal projection over the space $\mathcal{A}^k(\mathbf{U})$, which is given by the formula

$$F_k(z) = \langle F(w), K^k(z, w) \rangle_{\mathbf{A}^n(\mathbf{U})}.$$

Our formulas will be given in a form of differential operators which are reminiscent of the *Rodrigues formula*, a well known structure formula in the theory of classic orthogonal polynomials.

6.1. An orthogonal basis. Consider the functions Ψ_n^β , for every $n \geq 0$ and $\beta > 1$:

$$\Psi_n^\beta(z) = (2i)^{\beta+1} \frac{\Gamma(\beta+n)}{n!} \left(\frac{z-i}{z+i} \right)^n \left(\frac{1}{z+i} \right)^\beta.$$

It is well known that these functions constitute a basis of $A_{\beta-2}(\mathbf{U})$. A calculation using the special function formula

$$\int_0^\infty x^\alpha L_n^\alpha(x) e^{-xs} dx = \frac{\Gamma(\alpha+n+1)}{n!} s^{-\alpha-n-1} (s-1)^n$$

gives

$$Ber_\alpha l_n^{2\alpha-1} = \Psi_n^{2\alpha}.$$

Now write

$$\Psi_n(z) = \Psi_n^2(z) = (2i)^3 \frac{\Gamma(2+n)}{n!} \left(\frac{z-i}{z+i} \right)^n \left(\frac{1}{z+i} \right)^2$$

to denote a basis of $A(\mathbf{U})$, so that

$$Ber l_n^1 = \Psi_n.$$

Definition 4. Define a set of functions by

$$e_{n,m}(z) = \frac{1}{(2i)^n n!} \left(\frac{d}{dz} \right)^n [s^n \Psi_m(z)].$$

Proposition 4. The set $\{e_{k,m}\}_{k \geq 0, 0 \leq m < n}$ is an orthonormal basis of $\mathbf{A}^n(\mathbf{U})$.

Proof. Since

$$e_{n,k}(z) = Ber^n l_k^1 = s^{-1} W_{\Phi_n}(\mathcal{F}^{-1} l_k^1)(x, s),$$

the orthogonality follows from (2.2):

$$\begin{aligned} \langle e_{n,k}, e_{l,j} \rangle_{L^2(\mathbf{U}, dz)} &= \langle W_{\Phi_n}(\mathcal{F}^{-1} l_k^1), W_{\Phi_l}(\mathcal{F}^{-1} l_j^1) \rangle_{L^2(\mathbf{U}, s^{-2} dz)} \\ &= \langle l_n^0, l_l^0 \rangle_{L^2(\mathbb{R}^+)} \langle \mathcal{F}^{-1} l_k^1, \mathcal{F}^{-1} l_j^1 \rangle_{H^2(\mathbf{U})} \\ &= \delta_{n,l} \delta_{k,j}. \end{aligned}$$

The unitarity of Ber^n shows that, for every m , $\{e_{k,m}\}_{k \geq 0}$ spans $\mathcal{A}^m(\mathbf{U})$, since $\{l_n^1\}_{n \geq 0}$ spans $L^2(\mathbb{R}^+)$. From the decomposition (3.2), every element in $\mathbf{A}^n(\mathbf{U})$ can be written as a linear combination of elements of $\{\mathcal{A}^m(\mathbf{U})\}_{m < n}$. Therefore $\{e_{k,m}\}_{k \geq 0, 0 \leq m < n}$ spans $\mathbf{A}^n(\mathbf{U})$. \square

Corollary 1. The set $\{e_{k,m}\}_{0 \leq m < n}$ is an orthonormal basis of $\mathcal{A}^k(\mathbf{U})$.

Proof. This follows immediately from the decomposition (3.2). \square

6.2. The reproducing kernel. In our computations of the reproducing kernels we will need the following relation, which says essentially that the Bergman transform intertwines the representation of the affine group in $L^2(\mathbb{R}^+)$ with its representation in the Bergman space:

$$(6.1) \quad Ber(M_{-\mu} D_{\frac{1}{\eta}} f) = s^{-1} Ber(f) \left(\frac{z+u}{\eta} \right);$$

The identity (6.1) follows from the change of variables:

$$\int_0^\infty t^{\frac{1}{2}} e^{izt} (M_{-\mu} D_{\frac{1}{\eta}} f)(t) dt = \int_0^\infty t^{\frac{1}{2}} e^{i(\frac{z+u}{\eta})t} f(t) dt$$

It is interesting to observe that, defining a transform by $T = \text{Ber}(\mathcal{F}f)$ and $Mf(t) = t^{\frac{1}{2}}f(t)$, we have the following commutative diagram:

$$\begin{array}{ccc} H^2(\mathbf{U}) & \xrightarrow{T} & A^2(\mathbf{U}) \\ \mathcal{F} \downarrow & & \downarrow \mathcal{F} \\ L^2(\mathbb{R}^+) & \xrightarrow{M} & L^2(\mathbb{R}^+, t^{-1}dt) \end{array}$$

The Fourier isometry on the right column is the Paley-Wiener theorem for the Bergman space [9].

Now we compute the reproducing kernel of the wavelet space \mathcal{W}_{Φ_n} .

Theorem 3. *The reproducing kernel of \mathcal{W}_{Φ_n} is given by*

$$k^n(z, w) = \frac{1}{n!(2i)^n} \frac{s}{\eta} \left(\frac{d}{dz} \right)^n \left[s^n \Omega_n \left(\frac{z+u}{\eta} \right) \right],$$

where

$$\Omega_n(z) = 4(n+z-i) \left(\frac{1}{z+i} \right)^3 \left(\frac{z-i}{z+i} \right)^{n-1}.$$

Proof. The reproducing kernel of \mathcal{W}_{Φ_n} is

$$\begin{aligned} k^n(z, w) &= \langle T_{-\mu} D_{\eta} \Phi_n, T_{-x} D_s \Phi_n \rangle_{H^2(\mathbf{U})} \\ &= s \text{Ber}^n(M_{-\mu} D_{\frac{1}{\eta}} \mathcal{F} \Phi_n)(z) \\ &= \frac{s}{n!(2i)^n} \sum_{k=0}^n (2i)^k \binom{n}{k} \frac{s^k}{k!} \left(\frac{d}{dz} \right)^k \left[\text{Ber}(M_{-\mu} D_{\frac{1}{\eta}} \mathcal{F} \Phi_n)(z) \right] \\ &= \frac{s}{n!(2i)^n} \left(\frac{d}{dz} \right)^n \left[s^n \text{Ber}(M_{-\mu} D_{\frac{1}{\eta}} \mathcal{F} \Phi_n)(z) \right] \end{aligned}$$

Now, (6.1) gives

$$k^n(z, w) = \frac{1}{(2i)^n n!} \frac{s}{\eta} \left(\frac{d}{dz} \right)^n \left[s^n \text{Ber}(\mathcal{F} \Phi_n) \left(\frac{z+u}{\eta} \right) \right].$$

We just need to compute

$$\begin{aligned} \text{Ber}(\mathcal{F} \Phi_n) &= \int_0^\infty t l_n^0(2t) e^{itz} dt = \frac{1}{i} \frac{d}{dz} \int_0^\infty l_n^0(2t) e^{itz} dt \\ &= \frac{4}{i} \frac{d}{dz} \left[\left(\frac{z-i}{z+i} \right)^n \frac{1}{z+i} \right] = \Omega_n(z), \end{aligned}$$

and the formula is proved. \square

The next Lemma can be used to transfer properties from the spaces \mathcal{W}_{Φ_n} to the spaces $\mathcal{A}^n(\mathbf{U})$.

Lemma 1. *The operator*

$$\begin{aligned} E : \mathcal{W}_{\Phi_n} &\rightarrow \mathcal{A}^n(\mathbf{U}) \\ f &\rightarrow s^{-1} f(x, s), \end{aligned}$$

is unitary.

Proof. Clearly, E is isometric. Since l_m^1 is a basis of $L^2(\mathbb{R}^+)$, then $W_{\Phi_n}(\mathcal{F}^{-1} l_m^1)$ is a basis of \mathcal{W}_{Φ_n} . Then

$$E(W_{\Phi_n}(\mathcal{F}^{-1} l_m^1)) = s^{-1} W_{\Phi_n}(\mathcal{F}^{-1} l_m^1)(x, s) = \text{Ber}^n l_m^1(z) = e_{n,m}(z).$$

Thus, $E(\mathcal{W}_{\Phi_n})$ is dense in $\mathcal{A}^n(\mathbf{U})$. \square

Theorem 4. *The reproducing kernels of the spaces $\mathcal{A}^n(\mathbf{U})$, $K^n(z, w)$, are given by*

$$K^n(z, w) = \frac{1}{n!(2i)^n} \left(\frac{d}{dz} \right)^n \left[s^n \Omega_n \left(\frac{z+u}{\eta} \right) \right],$$

The reproducing kernels of the spaces, $\mathbf{A}^n(\mathbf{U})$, $\mathbf{K}^n(z, w)$, are given by

$$\mathbf{K}^n(z, w) = \sum_{k=0}^{n-1} \frac{1}{n!(2i)^n} \left(\frac{d}{dz} \right)^n \left[s^n \Omega_n \left(\frac{z+u}{\eta} \right) \right].$$

Proof. Let $f \in \mathcal{A}^n(\mathbf{U})$. Then, by the above Lemma, $sf(z) \in \mathcal{W}_{\Phi_n}$. Therefore,

$$sf(z) = \langle k^n(z, w), \eta f(w) \rangle_{\mathcal{W}_{\Phi_n}},$$

and

$$f(z) = \left\langle \frac{\eta}{s} k^n(z, w), f(w) \right\rangle_{\mathbf{A}^n(\mathbf{U})}.$$

We conclude that $K^n(z, w) = \frac{\eta}{s} k^n(z, w)$. The second assertion follows immediately from (4.9). \square

7. SAMPLING SEQUENCES AND WAVELET FRAMES

This section is devoted to sampling and frames for Wavelet frames and superframes. There are several approaches to general sampling and stability problems (see, for instance [11] and [12]), but we will follow mainly the one in [3], where, fixed an analyzing wavelet, the space of all continuous transforms (the "model space") is considered, in order to translate the frame problem in a sampling problem for such a model space.

Now, we will denote by $\Gamma(a, b)$ the set $\{z_{mk} = a^m(bk + i)\}$. We say that Γ is a *sampling sequence* for $\mathcal{A}^n(\mathbf{U})$ if there exist $A, B > 0$ such that, for every $F \in \mathcal{A}^n(\mathbf{U})$,

$$(7.1) \quad A \|F\|_{\mathcal{A}^n(\mathbf{U})}^2 \leq \sum_{z \in \Gamma(a, b)} s^2 |F(z)|^2 \leq B \|F\|_{\mathcal{A}^n(\mathbf{U})}^2.$$

We say that $\mathcal{W}(\psi, \Gamma(a, b))$ is a wavelet frame for $H^2(\mathbf{U})$ if

$$(7.2) \quad A \|f\|_{H^2(\mathbf{U})}^2 \leq \sum_{j, k} |\langle f, T_{a^j b k} D_{a^j} \psi \rangle|^2 \leq B \|f\|_{H^2(\mathbf{U})}^2.$$

Since

$$\langle f, T_x D_s \Phi_n \rangle_{H^2(\mathbf{U})} = W_{\Phi_n} f(x, s) = s \text{Ber}^n \mathcal{F}f(z),$$

it is plain that Γ is a *sampling sequence* for $\mathcal{A}^n(\mathbf{U})$ if and only if $\mathcal{W}(\psi, \Gamma(a, b))$ is a wavelet frame for $H^2(\mathbf{U})$. Our next result is an upper bound on the size of the parameters (a, b) (or a lower bound on density) necessary to generate sampling sequences in the true polyanalytic Bergman space, or wavelet frames with Laguerre functions.

The vector valued system $\mathcal{W}(\mathbf{g}, \Lambda) = \{T_{a^j b k} D_{a^j} \mathbf{g}\}$ is a *wavelet superframe* for \mathcal{H} if there exist constants A and B such that, for every $\mathbf{f} \in \mathcal{H}$,

$$(7.3) \quad A \|\mathbf{f}\|_{\mathcal{H}}^2 \leq \sum_{j, k} |\langle \mathbf{f}, T_{a^j b k} D_{a^j} \mathbf{g} \rangle_{\mathcal{H}}|^2 \leq B \|\mathbf{f}\|_{\mathcal{H}}^2.$$

Superframes were introduced in a more abstract form in [17] and in the context of "multiplexing" in [4]. Take the analyzing vector to be $\Phi_n = (\Phi_0, \dots, \Phi_{n-1})$. Using the identity

$$\langle \mathbf{f}, D_s T_x \Phi_n \rangle_{\mathcal{H}} = s \mathbf{B} \mathbf{e}^n \mathbf{f},$$

we see that $\mathcal{W}(\Phi_n, \Lambda)$ is a wavelet superframe for \mathcal{H} if and only if Γ is a sampling sequence for $\mathbf{A}^n(\mathbf{U})$.

7.1. Existence of sampling sequences and frames. In this subsection we will prove the existence of wavelet frames with the functions Φ_n , provided the hyperbolic lattice is sufficiently dense. We need some notations from [3]. First recall the hyperbolic distance in the half-plane

$$d(z_1, z_2) = \frac{1}{2} \log \frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)},$$

where $\rho(z_1, z_2)$ is the pseudohyperbolic distance:

$$\rho(z_1, z_2) = \left| \frac{z_1 - z_2}{z_1 - \bar{z}_2} \right|.$$

Given a continuous function h in $\mathbb{R} \times \mathbb{R}^+$, we define its local maximal function as

$$Mh(z) = \sup_{w \in B(z, 1)} |h(w)|,$$

where $B(z, 1)$ is the hyperbolic ball of center z and radius 1 in $\mathbb{R} \times \mathbb{R}^+$. Now, define

$$k_g(z) = \langle \psi, T_x D_s g \rangle_{H^2(\mathbf{U})}, \quad z = x + is,$$

and

$$MB = \{g : Mk_g \in L^1(\mathbf{U})\}$$

With these notations, Theorem 4.9 in [3] reads:

Theorem B. *Let W_g be the wavelet space associated to the analyzing wavelet g . If $g \in MB$, then there is a δ such that every uniformly discrete set Γ satisfying $B(z, \delta) \cap \Gamma \neq \emptyset$ for every z , is a sampling set for W_g .*

Lemma 2. $\Phi_n(z) \in MB$.

Proof. Using (5.2) twice and evaluating the resulting wavelet transforms on the Fourier side, gives:

$$\begin{aligned} \langle \Phi_n, T_x D_s \Phi_n \rangle_{H^2(\mathbf{U})} &= \sum_{k=0}^n \sum_{j=0}^n (-2)^{k+j} \binom{n}{k} \binom{n}{j} \left\langle \psi_{j+\frac{1}{2}}, T_x D_s \psi_{k+\frac{1}{2}} \right\rangle_{H^2(\mathbf{U})} \\ &= \sum_{k=0}^n \sum_{j=0}^n (-2)^{k+j} \binom{n}{k} \binom{n}{j} s^{k+1} \int_0^\infty t^{1+k+j} e^{(iz-1)t} dt \\ &= \sum_{k=0}^n \sum_{j=0}^n i^{1+k+j} (-2)^{k+j} \binom{n}{k} \binom{n}{j} \Gamma(1+k+j) \frac{s^{k+1}}{(z+i)^{1+k+j}} \end{aligned}$$

Now, since $u_{k,j}(z) = 1/(z+i)^{1+k+j}$ is analytic on the upper half plane, then its maximum on the ball $B(z, \delta)$ is bounded by the average of $u_{k,j}(z)$ on the ball. For this reason, $u_{k,j}(z) \in L^1(\mathbf{U})$ implies $u_{k,j}(z) \in MB$. As a result, also $\Phi_n(z) \in MB$. \square

Combining this Lemma with the theorem above, we can assure the existence of Wavelet frames with windows Φ_n .

Theorem 5. *It is possible to choose a $\delta > 0$ such that every uniformly discrete set Γ satisfying $B(z, \delta) \cap \Gamma \neq \emptyset$ for every z , $\mathcal{W}(\Phi_n, \Lambda)$ is a wavelet frame for $H^2(\mathbf{U})$.*

Corollary 2. *It is possible to choose a δ such that every uniformly discrete set Γ satisfying $B(z, \delta) \cap \Gamma \neq \emptyset$ for every z , Γ is a sampling sequence for $\mathcal{A}^n(\mathbf{U})$.*

Corollary 3. *It is possible to choose a δ such that every uniformly discrete set Γ satisfying $B(z, \delta) \cap \Gamma \neq \emptyset$ for every z , $\mathcal{W}(\Phi_n, \Lambda)$ is a wavelet superframe for $H^2(\mathbf{U})$.*

Proof. The arguments we have used in Lemma 2 can be adapted to prove the existence of superframes of this form, since

$$\langle \Phi_n, T_x D_s \Phi_n \rangle_{\mathcal{H}} = \sum_{k=0}^{n-1} \langle \Phi_k, T_x D_s \Phi_k \rangle_{H^2(\mathbf{U})} \in MB.$$

□

7.2. Necessary conditions on the hyperbolic lattice. Now we give necessary conditions for the sampling sequences and frames to exist. Our next result is an upper bound on the size of the parameters a and b (or a lower bound on density) necessary to generate sampling sequences in the true polyanalytic Bergman space, or wavelet frames with the functions Φ_n . For this purpose, we will adapt the proof of the necessity part of Theorem 1.1 in [28]. An essential step is to associate to $\Gamma(a, b)$ the analytic function

$$h(z) = \left(\prod_{k=0}^{\infty} \frac{\sin \pi b^{-1} a^{-k} (ia^k - z)}{\sin \pi b^{-1} a^{-k} (ia^k + z)} \right) \left(\prod_{m=1}^{\infty} e^{\frac{2\pi}{b}} \frac{\sin \pi b^{-1} a^m (z - ia^{-m})}{\sin \pi b^{-1} a^m (z + ia^{-m})} \right),$$

which vanishes in $\Gamma(a, b)$ and plays the role of the sine function in the Paley-Wiener space and of the Weierstrass σ -function in the Bargmann-Fock space. Following [28], one can check that

$$h(az) = -e^{-\frac{2\pi}{b}} h(z).$$

Then, from an estimate of the growth of h on the strip $a^{-1/2} < y < a^{1/2}$, the following global estimate is obtained:

$$(7.4) \quad |h(z)| \leq Cs^{-\frac{2\pi}{b \ln a}}.$$

Another ingredient in the proof is a result from [3], which gives the stability, with respect to the jittered error, for general wavelet spaces.

Theorem C [3, Theorem 4.4]. *Let W_g be the wavelet space associated to the analyzing wavelet g . If $\Lambda = \{z_j\}$ is a sampling set for W_g , there exists $\delta > 0$ such that if $\Gamma = \{w_j\}$ satisfies $\rho(z_j, w_j) < \delta$ for all j , then Γ is also a sampling set.*

Theorem 6. *If $z_{mk} = a^m(bk + i)$ is a sampling sequence for $\mathcal{A}^n(\mathbf{U})$, then*

$$b \log a < 2\pi(n + 1).$$

Proof. Suppose that $\Gamma(a, b)$ is a sampling sequence. The growth estimate (7.4) gives:

$$h(z) \in A(\mathbf{U}) \iff \frac{4\pi}{b \ln a} < 1.$$

As a result, if $b \log a > 4\pi(n+1)$, then $h^{n+1}(z) \in A(\mathbf{U})$. Thus, there exists a $f \in L^2(\mathbb{R}^+)$ such that $h^{n+1}(z) = \text{Ber } f(z)$. Now consider the function

$$H(z) = \frac{1}{(2i)^n n!} \left(\frac{d}{dz} \right)^n [s^n h^{n+1}(z)].$$

Clearly,

$$H(z) = \frac{1}{(2i)^n n!} \left(\frac{d}{dz} \right)^n [s^n \text{Ber } f(z)] = \text{Ber}^n f(z) \in \mathcal{A}^n(\mathbf{U}).$$

Since $H(z)$ vanishes on $\Gamma(a, b)$, $\Gamma(a, b)$ cannot be a sampling sequence for $\mathcal{A}^n(\mathbf{U})$. It follows that $b \log a \leq 2\pi(n+1)$.

To prove that the inequality is strict, observe that, by Theorem C, there exists a $\delta > 0$ such that if $\Gamma = \{w_{mk}\}$ satisfies $\rho(z_{mk}, w_{mk}) < \delta$ for all m, k , then Γ is also a sampling sequence. Thus, if $b \log a = 2\pi(n+1)$, we can choose δ_0 such that $w_{mk} = a^m(bk + i(1 - \delta_0))$ satisfies $\rho(z_{mk}, w_{mk}) < \delta$ and therefore it is a sampling sequence. This is impossible by the argument in the previous paragraph, since $\{w_{mk}\} = \Gamma(a, b/(1 - \delta_0))$ and $b/(1 - \delta_0) \log a > b \log a = 2\pi(n+1)$. \square

Corollary 4. *If $\mathcal{W}(\Phi_n, \Lambda)$ is a wavelet frame in $H^2(\mathbf{U})$, then*

$$b \log a < 2\pi(n+1).$$

Proof. The equivalence between the wavelet frame condition and the sampling condition in the true polyanalytic Bergman space follows from the identity:

$$\langle f, D_s T_x \Phi_n \rangle_{H^2(\mathbf{U})} = W_{\Phi_n} f(x, s) = s \text{Ber}^n \mathcal{F}f(z),$$

from where it is easily seen that (1.1) and (7.2) are equivalent. \square

Remark 1. *Minor adaptations in the proofs of this paper allow us to introduce a weight in the spaces, by considering the Bergman norm*

$$\int_{\mathbf{U}} |f(z)|^2 s^\alpha dx ds < \infty,$$

with $\alpha > -1$. In this case, Λ is a sampling sequence in the space $\mathcal{A}_\alpha^n(\mathbf{U})$ if and only if $\mathcal{W}(\Phi_n^\alpha, \Lambda)$ is a wavelet frame in $H^2(\mathbf{U})$, where Φ_n^α is defined by $\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{1}{2}} 1_n^{2\alpha}(2t)$. A necessary condition for this to happen is:

$$b \log a < 2\pi \frac{n+1}{\alpha+1}.$$

Remark 2. *Since the superframe property requires every system $\mathcal{W}(\Phi_k, \Lambda)$ to be a frame, it follows that $b \log a < 2\pi$ is a necessary condition for $\mathcal{W}(\Phi_n, \Lambda)$ to be a wavelet superframe.*

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